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NONUNIQUENESS FOR SOLUTIONS OF THE KORTEWEG-DeVRIES EQUATION

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ABSTRACT. Variants of the inverse scattering method give examples of nonuniqueness for the Cauchy problem for KdV. One example gives a nontrivial C^∞ solution u in a domain $\{(x,t): 0 < t < H(x)\}$ for a positive nondecreasing function H, such that u vanishes to all orders as $t \downarrow 0$. This solution decays rapidly as $x \to +\infty$, but cannot be well behaved as x moves left. A different example of nonuniqueness is given in the quadrant $x \geq 0, t \geq 0$, with nonzero initial data.

1. INTRODUCTION AND SUMMARY

The initial value problem

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.1}$$

$$u(x,0) = U_0(x) \tag{1.2}$$

for the Korteweg-deVries equation specifies data on a characteristic line. Thus it is natural to look for nonuniqueness, possibly associated with poor behavior as $|x| \to +\infty$. Here we present two nonuniqueness results.

First, we construct a C^∞ function u_1 such that

(i) u_1 solves KdV in a neighborhood of the form $\mathscr{U}_1 = \{(x,t): 0 < t < h(x)\}$ for positive nondecreasing h with $\lim_{x\to+\infty} h(x) = +\infty$.

(ii) $u_1(x,0) = 0$ for all x.

(iii) u_1 cannot vanish identically in $\mathscr{U}_1$.

The domain $\mathscr{U}_1$ can be made arbitrarily big in the sense that, given any point (x_0, t_0) with $t_0 > 0$, we can find such u_1 and $\mathscr{U}_1$ with $(x_0, t_0) \in \mathscr{U}_1$. This construction can be adapted to produce two different solutions to KdV arising from any reasonably nice choice of initial data.

While the first method treats initial data given on the full line at $t = 0$, it gives a solution which is not global in time. The second method treats initial data on a half-line $x \geq x_0$, but produces solutions for all $t > 0$. More precisely

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we construct distinct solutions to KdV in $Q = \{(x, t): x_0 < x < \infty, 0 < t < \infty\}$ which agree at $t = 0$ on $x_0 < x < \infty$.

Both constructions use variants of the inverse scattering method introduced by Gardner, Greene, Kruskal, and Miura [6]. This method exploits the relationship between KdV and the linear Airy equation

$$\Omega_t + \Omega_{xxx} = 0. \tag{1.3}$$

Tanaka's argument in [13] proves the following result.

Theorem T. *Let $\mathscr{D}$ be a domain of the form*

$$\mathscr{D} = \{(x, t): 0 < t < h(x), x \in \mathbf{R}\},$$

where h is nondecreasing and valued in $[0, \infty]$. Suppose that Ω is a C^∞ solution of (1.3) in $\mathscr{D}$ and that all derivatives of Ω decay rapidly as $x \to +\infty$. Suppose that for each (x, t) in $\mathscr{D}$, $B(x, \cdot t)$ solves the Marchenko equation

$$B(x, y, t) + \Omega(x + y, t) + \int_0^\infty \Omega(x + y + z, t) B(x, z, t)\, dz = 0. \tag{1.4}$$

Then the function u defined by $u(x, t) = -\partial_x B(x, 0, t)$ solves KdV in $\mathscr{D}$.

Tanaka proved Theorem T for $\mathscr{D} = \mathbf{R} \times \mathbf{R}^+$, i.e. for $h(x) \equiv +\infty$,but it is straightforward to check that his proof carries over to these more general domains $\mathscr{D}$.

The crucial efforts in this paper are to obtain the appropriate solutions Ω of (1.3) and to show that the corresponding Marchenko equations (1.4) have solutions. These efforts are easy to outline.

We begin with a special case of Hörmander's Theorem 5.2.5 in [8] to obtain a real-valued function $\Omega^\#(x, t)$ with the following properties:

$$\Omega^\# \in C^\infty(\mathbf{R} \times \mathbf{R}^+), \tag{1.5.a}$$

$$\Omega^\#_t + \Omega^\#_{xxx} = 0 \quad \text{in } \mathbf{R} \times \mathbf{R}^+, \tag{1.5.b}$$

$$\Omega^\#(x, t) \equiv 0 \quad \text{for } t \le 0, \tag{1.5.c}$$

$$(0, 0) \in \operatorname{supp}(\Omega^\#); \text{ indeed, there is a } t_1 > 0 \text{ such that } \Omega^\#(0, t) \ne 0 \text{ whenever } 0 < t < t_1. \tag{1.5.d}$$

One can verify that $\Omega^\#$ and all its derivatives decay faster than exponentially as $x \to +\infty$ for fixed positive t, and that $\Omega^\#$ and all its derivatives decay to the zero function as $t \downarrow 0$ both in L^∞ and L^1 on each half-line $[x_0, \infty)$.

Results 1. *Nonuniqueness for* (1.1), (1.2) *with* $u_0 \equiv 0$ *on* $\mathbf{R}$. Pick (x_0, t_0) with $t_0 > 0$. For a positive parameter ε to be chosen later, let $\Omega(x, t; \varepsilon) = \varepsilon\Omega^\#(x, t)$ and consider the Marchenko equation

$$B(x, y, t) + \Omega(x + y, t; \varepsilon) + \int_0^\infty \Omega(x + y + z, t; \varepsilon) B(x, z, t)\, dz = 0. \tag{1.6}$$

The kernel $\Omega(x, t; \varepsilon)$ is not the usual one suggested by the inverse scattering method for (1.1), (1.2) with $u_0 \equiv 0$: the classical inverse scattering construction would use $\Omega \equiv 0$ since the reflection coefficient for the potential $u_0 \equiv 0$ is identically 0. To solve (1.5) one estimates $\Omega^{\#}$ carefully and then chooses ε artfully to make sure that the integral operator defined by the kernel $\Omega(x, t; \varepsilon)$ with $(x, t) = (x_0, t_0)$ has norm less than 1, and then finds $h(x)$ so all integral operators with $0 < t < h(x)$ also have operator norm less than 1. Let $B_1(x, y, t)$ denote the solution to (1.6); set $u_1(x, t) = -\partial_x B_1(x, 0, t)$. By Theorem T u_1 solves KdV in $\mathscr{U}_1$. Analysis of Ω shows that u_1 and all its derivatives approach 0 as t decreases to 0. By using the relation

$$B_{xx} - B_{xy} = u(x, t)B,$$

called the "wave equation" by Deift and Trubowitz [3], one shows that if $u \equiv 0$ in $\mathscr{U}_1$, then $\Omega(x, t; \varepsilon) \equiv 0$ in $\mathscr{U}_1$, which contradicts (1.5.d). Details of this argument are given in §2.

In §3 we discuss the further application of this construction to nonuniqueness for (1.1), (1.2) with more general initial data u_0.

Result 2. *Nonuniqueness for all* $t > 0$ *in* $x \geq x_0$. Since KdV is invariant under translation in x, it suffices to work in

$$Q_0 = \{(x, t): 0 < t < \infty, 0 < x < \infty\}.$$

Careful analysis of $\Omega^{\#}$ gives us positive constants K and τ such that

$$\Omega_2(x, t) := \Omega^{\#}(x, t) + 2Ke^{t\tau - x\tau^{1/3}} > 0$$

in Q_0. This Ω_2 solves the Airy equation, is smooth, and decays fast as $x \to +\infty$. Consider the Marchenko equation

$$(1.6.2) \quad B(x, y, t) + \Omega_2(x + y, t) + \int_0^{\infty} \Omega_2(x + y + z, t)B(x, z, t)\, dz = 0.$$

If $(x, t) \in Q_0$, the integral operator is symmetric and positive in $L^2(\mathbf{R}^+)$; thus (1.5.2) can be solved. Let B_2 denote the solution, and set $u_2(x, y) = -\partial_x B_2(x, 0, t)$. Theorem T tells us that u_2 solves KdV in Q_0. Next set

$$\Omega_3(x, t) := 2Ke^{t\tau - x\tau^{1/3}}$$

and consider the Marchenko equation (1.6.3), which is (1.6) with Ω_3 in place of Ω. As before we can solve (1.6.3) to get B_3, set $u_3 =: -\partial_x B_3(x, 0, t)$, and use Theorem T to see that u_3 solves KdV. Since $\Omega^{\#}(x, 0) \equiv 0$, we see that $B_2(x, 0, 0) = B_3(x, 0, 0)$ in $x > 0$, whence $u_2(x, 0) = u_3(x, 0)$ in $x > 0$. In §4 we present details of this argument and also show that u_2 cannot be identically equal to u_3 in Q_0.

It is not clear whether our solution u_1 can be extended to a solution of KdV in any strip $\{(x, t): 0 < t < T\}$. If any such extension were possible, it could not evolve in any of the uniqueness classes described by Lax [11], Temam [14], Menikoff [12], Kruzhkov and Faminskii [10], and others. Clearly the L^2 norm could not be a conserved quantity for such an extended u_1. Either u_1 cannot exist all the way to the left, or it blows up as $x \to -\infty$. The behavior of $\Omega^{\#}$ as $x \to -\infty$ is discussed in §3, as are some speculations about u_1 as $x \to -\infty$.

Most notation will be standard. The following definitions have been used in our previous papers and should be made explicit here:

$L^1_N(\mathbf{R}) =: \{f: \int_{-\infty}^{\infty} |f(x)|(1+|x|)^N\,dx < \infty\}$ for $N \geq 1$.

$L^1_N(\mathbf{R}^+) =: \{f: \int_0^{\infty} |f(x)|(1+|x|)^N\,dx < \infty\}$ for $N \geq 1$.

$L^p(+\infty) =: \{f: f \in L^p([a, +\infty))$ for all finite $a\}$.

$\|\ \|_{\mathrm{op}, L^p(\mathbf{R}^+)}$ denotes the operator norm in $\mathscr{L}(L^p(\mathbf{R}^+), L^p(\mathbf{R}^+))$.

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2. The first nonuniqueness result

Theorem 2.1. *Choose any point* (x_0, t_0) *with* $t_0 > 0$. *There is a function* u_1 *and a domain* $\mathscr{U}_1$ *such that*

(a) $\mathscr{U}_1$ *has the form* $\{(x,t): 0 < t < h(x), x \in \mathbf{R}\}$ *for a positive nondecreasing function* h *with* $\lim_{x\to+\infty} h(x) = +\infty$.

(b) $(x_0, t_0) \in \mathscr{U}_1$.

(c) u_1 *is a* C^∞ *solution of KdV in* $\mathscr{U}_1$.

(d) u_1 *and all its derivatives approach* 0 *as* $t \downarrow 0$.

(e) u_1 *is not identically zero in* $\mathscr{U}_1$.

Proof. The rest of this section in devoted to this proof. It begins with Hörmander's nonuniqueness theorem for the Airy equation.

A. The Airy equation

Following Hörmander's Theorem 5.2.5 of [8] define a function $\Omega^{\#}$ by

$$\Omega^{\#}(x,t) =: \int_{i\tau-\infty}^{i\tau+\infty} e^{i\{xr(s)-ts\}} e^{-(s/i)^{2/3}}\,ds, \tag{2.1}$$

where $\tau > 0$ and $r(s) = (-s)^{1/3}$. To be precise we take

$$r(s) = |s|^{1/3} e^{i(\pi + \operatorname{Arg} s)/3} \quad \text{with } 0 < s < \pi$$

and

$$(s/i)^{2/3} = |s|^{2/3} e^{i2(\operatorname{Arg}(s/i))/3} \quad \text{with } -\pi/2 < \operatorname{Arg}(s/i) < \pi/2,$$

since $s = \sigma + i\tau$ with $\sigma \in \mathbf{R}$ and $\tau > 0$. With these choices, Hörmander's theorem tells us that

(2.2a) $\Omega^{\#}$ is independent of τ,

(2.2b) $\Omega^{\#}$ is C^{∞} in $\mathbf{R} \times \mathbf{R}$,

(2.2c) $\Omega^{\#}$ solves $\Omega_t + \Omega_{xxx} = 0$ in $\mathbf{R} \times \mathbf{R}$,

(2.2d) $\Omega^{\#}(x, t) = 0$ whenever $t \leq 0$,

(2.2e) $(0, 0) \in \operatorname{supp}(\Omega^{\#})$; indeed there is a positive t_1 such that

$$\Omega^{\#}(0, t) \neq 0 \text{ whenever } 0 < t < t_1 .$$

One can verify that

(2.2f) $\Omega^{\#}$ is real-valued

since the integrand at $s = -\sigma + i\tau$ is the conjugate of the integrand at $s = \sigma + i\tau$. Further note that $\Omega^{\#}(z, t)$ is analytic in z near the real-z-axis. Thus it follows from (2.2e) that

(2.2g) there is a positive t_1 such that if $0 < t < t_1$, $\Omega^{\#}(x, t)$ cannot vanish identically on any half-line $[x_0, +\infty)$.

We now study the decay of $\Omega^{\#}$ as $x \to +\infty$, and its convergence to 0 as $t \downarrow 0$.

Lemma 2.2. *Keep $\tau > \tau_0 = (2/\sqrt{3})^3$ as (2.1). Then for all $x \geq 0$ and $t \geq 0$*

$$|\Omega^{\#}(x, t)| \leq K_1(\tau) e^{t\tau - x},$$

where $K_1(\tau) = \int_{i\tau-\infty}^{i\tau+\infty} e^{-|s|^{2/3}/2} ds$. Note that $K_1(\tau) \to 0$ as $\tau \to +\infty$.

Proof. From (2.1)

$$|\Omega^{\#}(x, t)| \leq e^{t\tau} \int_{i\tau-\infty}^{i\tau+\infty} e^{-x \operatorname{Im} r(s)} e^{-\operatorname{Re}((s/i)^{2/3})} ds .$$

Note that

$$|s|^{1/3} \frac{\sqrt{3}}{2} \leq \operatorname{Im}(r(s)) = |s|^{1/3} \sin\left(\frac{\pi + \operatorname{Arg} s}{3}\right) \leq |s|^{1/3}$$

since $0 < \operatorname{Arg} < \pi$, and that

$$\frac{1}{2}|s|^{2/3} \leq \operatorname{Re}((s/i)^{2/3}) = |s|^{2/3} \cos\left(\frac{2 \operatorname{Arg}(s/i)}{3}\right) \leq |s|^{2/3}$$

since $-\pi/2 < \operatorname{Arg}(s/i) < \pi/2$. Keep $x \geq 0$. Now

$$|\Omega^{\#}(x, t)| \leq e^{t\tau} \int_{i\tau-\infty}^{i\tau+\infty} e^{-x|s|^{1/3}\sqrt{3}/2} e^{-|s|^{2/3}/2} ds .$$

Since $|s| \geq \tau \geq (2/\sqrt{3})^3$, we get

$$|\Omega^{\#}(x,t)| \leq e^{t\tau - x} \int_{i\tau-\infty}^{i\tau+\infty} e^{-|s|^{2/3}/2}\, ds. \quad \square$$

Lemma 2.3. *Suppose* $x_1 < 0$. *Keep* $\tau > \tau_1 = 8(|x_1| + 1)^3$ *as in* (2.1). *If* $x_1 < x \leq 0$ *and* $t > 0$, *then*

$$|\Omega^{\#}(x,t)| \leq e^{t\tau} K_2(\tau),$$

where $K_2(\tau) = \int_{i\tau-\infty}^{i\tau+\infty} e^{-|s|^{1/3}}\, ds$. *Note that* $K_2(\tau) \to 0$ *as* $\tau \to +\infty$.

Proof. From (2.1)

$$|\Omega^{\#}(x,t)| \leq e^{t\tau} \int_{i\tau-\infty}^{i\tau+\infty} e^{-x\,\mathrm{Im}(r(s))} e^{-\mathrm{Re}((s/i)^{2/3})}\, ds.$$

As before $(\sqrt{3}/2)|s|^{1/3} \leq \mathrm{Im}(r(s)) \leq |s|^{1/3}$ and $(1/2)|s|^{2/3} \leq \mathrm{Re}((s/i)^{2/3}) \leq |s|^{2/3}$. Here $x \leq 0$, so

$$|\Omega^{\#}(x,t)| \leq e^{t\tau} \int_{i\tau-\infty}^{i\tau+\infty} e^{|s|^{1/3}(|x| - |s|^{1/3}/2)}\, ds.$$

Since $\tau > \tau_1 = 8(|x_1|+1)^3$ and $|x| < |x_1|$, we get $|x| - |s|^{1/3}/2 < |x_1| - \tau^{1/3}/2 < -1$. Thus

$$|\Omega^{\#}(x,t)| \leq e^{t\tau} \int_{i\tau-\infty}^{i\tau+\infty} e^{-|s|^{1/3}}\, ds. \quad \square$$

Corollary 2.4. *Fix* x_0 *in* $\mathbf{R}$. *Choose* τ_0 *and* τ_1 *as above. For* $\tau \geq \max\{\tau_0, \tau_1\}$

$$|\Omega^{\#}(x,t)| \leq e^{t\tau}[K_1(\tau) + K_2(\tau)]e^{-x} \quad \textit{whenever } x \geq x_0.$$

Proof. This follows from the previous two lemmas. $\square$

Corollary 2.5. *Let* ν *be a positive integer,* $x_0 \in \mathbf{R}$, *and* $t \geq 0$. *Keep* $\tau \geq \tau_0$. *There is a constant* $M_\nu(x_0, \tau)$ *such that*

$$(2.3) \qquad |\partial_x^\nu \Omega^{\#}(x,t)| \leq M_\nu(x_0,\tau) e^{t\tau} e^{-x} \quad \textit{whenever } x \geq x_0 \textit{ and } 0 \leq t \leq t_0.$$

Proof. By (2.1)

$$\partial_x^\nu \Omega^{\#}(x,t) = \int_{i\tau-\infty}^{i\tau+\infty} (ir(s))^\nu e^{i\{xr(s) - ts\}} e^{-(s/i)^{2/3}}\, ds.$$

So for $x > 0$

$$|\partial_x^\nu \Omega^{\#}(x,t)| \leq \int_{i\tau-\infty}^{i\tau+\infty} |s|^{\nu/3} e^{-x} e^{t\tau} e^{-|s|^{2/3}/2}\, ds$$
$$\leq e^{t\tau} e^{-x} \int_{i\tau-\infty}^{i\tau+\infty} |s|^{\nu/3} e^{-|s|^{2/3}/2}\, ds.$$

The integral defines $M_\nu(x_0, \tau)$ for all nonnegative x_0. For $x < 0$, one appeals to the case $x = 0$ and to the continuity of $\partial_x^\nu \Omega^{\#}$. $\square$

By Hörmander's Theorem 5.2.5 [8] we know that $\partial_x^\nu \Omega(x, 0) = 0$ for all x and all orders ν. We need to see how $\partial_x^\nu \Omega^{\#}(x,t)$ approaches 0 as $t \downarrow 0$.

Lemma 2.6. *Pick $\nu \geq 0$ and $x_0 \in \mathbf{R}$. Then as $t \downarrow 0$*

$$\partial_x^\nu \Omega^\#(x,t) \to \partial_x^\nu \Omega^\#(x,0) = 0 \quad in\ L^\infty([x_0,\infty)), \tag{2.4a}$$

$$\partial_x^\nu \Omega^\#(x,t) \to \partial_x^\nu \Omega^\#(x,0) = 0 \quad in\ L^1([x_0,\infty)), \tag{2.4b}$$

$$\partial_x^\nu \Omega^\#(x,t) \to \partial_x^\nu \Omega^\#(x,0) = 0 \quad in\ L^2([x_0,\infty)). \tag{2.4c}$$

Proof. Since $\Omega^\#$ is C^∞ in $\mathbf{R}\times\mathbf{R}$ it suffices to treat $x_0 = 0$. (If $x_0 < 0$, then $\partial_x^\nu \Omega^\#(x,t) \to \partial_x^\nu \Omega^\#(x,0) = 0$ uniformly in $[x_0, 0]$.) Keep $x \geq 0$ and $\tau \geq \tau_0$. Now

$$\partial_x^\nu \Omega^\#(x,t) - \partial_x^\nu \Omega^\#(x,0) = \int_{i\tau-\infty}^{i\tau+\infty} (ir(s))^\nu e^{ixr(s)}\{e^{-its} - 1\} e^{-(s/i)^{2/3}}\, ds.$$

So

$$\begin{aligned} &|\partial_x^\nu \Omega^\#(x,t) - \partial_x^\nu \Omega^\#(x,0)| \\ &\quad \leq \int_{i\tau-\infty}^{i\tau+\infty} |s|^{\nu/3} e^{-x\,\mathrm{Im}(r(s))} |e^{-its} - 1| e^{-|s|^{2/3}/2}\, ds \\ &\quad \leq e^{-x} \int_{i\tau-\infty}^{i\tau+\infty} |s|^{\nu/3} |e^{its} - 1| e^{-|s|^{2/3}/2}\, ds. \end{aligned}$$

Recall that $\tau \geq \tau_0$ and $x \geq 0$ imply that $-x\,\mathrm{Im}(r(s)) \leq -x$ for all s with $\mathrm{Im}\, s = \tau$. Applying the Lebesgue dominated convergence theorem we see that

$$|\partial_x^\nu \Omega^\#(x,t) - \partial_x^\nu \Omega^\#(x,0)| \leq e^{-x} Q_\nu(t,\tau),$$

where $Q_\nu(t,\tau) \to 0$ as $t \to 0$. Results (2.4a) and (2.4b) now follow immediately, and (2.4c) follows from (2.4a) and (2.4b). □

B. Analysis of the Marchenko equation

For a positive parameter ε to be chosen later, let

$$\Omega(x,t) = \Omega(x,t;\varepsilon) = \varepsilon\Omega^\#(x,t).$$

Consider the Marchenko equation

$$B(x,y,t) + \Omega(x+y,t;\varepsilon) + \int_0^\infty \Omega(x+y+z,t;\varepsilon) B(x,z,t)\, dz = 0 \tag{2.5}$$

for parameter points (x,t) in $\mathbf{R}\times\mathbf{R}^+$ are variable y in $\mathbf{R}^+$.

In the classical inverse scattering construction of Faddeev [5] one shows that (2.5) has a solution $B(x,\cdot t)$ for each parameter pair (x,t) because of the form of the kernel as a sum of two terms: one, the inverse Fourier transform of a function (the reflection coefficient) which is almost everywhere smaller than 1 in absolute value; the other, a positive linear combination of decaying exponentials. In this paper the kernel Ω does not have this form, and we need a different proof of solvability for (2.5).

For $x \in \mathbf{R}$ and $t > 0$ let Ω_x^t denote the operator in $\mathscr{L}(L^2(\mathbf{R}^+), L^2(\mathbf{R}^+))$ given by

$$\Omega_x^t[g](y) = \int_0^\infty \Omega(x+y+z, t) g(z)\, dz.$$

Let ω_x^t be the function defined by

$$\omega_x^t(y) = \begin{cases} \Omega(x+y, t) & \text{if } y \geq 0, \\ 0 & \text{if } y \leq 0. \end{cases}$$

By (2.3) we see that $\omega_x^t \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$. For $g \in L^2(\mathbf{R}^+)$ let

$$\tilde{g}(z) = \begin{cases} 0 & \text{if } z \geq 0, \\ g(-z) & \text{if } z < 0. \end{cases}$$

It follows that

$$\Omega_x^t[g] = \omega_x^t * \tilde{g}, \tag{2.6}$$

where $*$ denotes convolution. One notes that (2.6) implies

$$||\Omega_x^t||_{\text{op}} \leq ||\omega_x^t||_{L^1(\mathbf{R})} = \int_{s=x}^\infty |\Omega(s, t)|\, ds,$$

where $||\cdot||_{\text{op}}$ denotes the operator norm on $L^2(\mathbf{R}^+)$. Equation (2.5) is now equivalent to

$$(I + \Omega_x^t)[B(x, \cdot, t)] = -\omega_x^t.$$

Since Ω_x^t is selfadjoint and compact, the existence of $(I + \Omega_x^t)^{-1}$ would follow from a proof that $I + \Omega_x^t$ is one-to-one in $L^2(\mathbf{R}^+)$. Such a proof for arbitrary (x, t) would have to rely on very specific properties of the kernel in Ω_x^t. It is not sufficient to use only the fact that $\Omega(x, t)$ decays at least exponentially as $x \to +\infty$: the equation

$$g(y) - \int_0^\infty 2\alpha e^{-\alpha(y+z)} g(z)\, dz = 0$$

has nontrivial solution $g(y) = e^{-\alpha y}$ when $\alpha > 0$.

Proposition 2.7. *Given any* (x_0, t_0) *with* $t_0 > 0$, *there is a domain* $\mathscr{U}_1$ *of the form required by Theorem* 2.1(a) *containing* (x_0, t_0) *such that the* $L^2(\mathbf{R}^+)$-*operator norm of* Ω_x^t *is less than* 1 *on* $\mathscr{U}_1$.

Proof in the case $x_0 \geq 0$. Recall that

$$||\Omega_x^t||_{\text{op}} \leq \varepsilon \int_0^\infty |\Omega^\#(s, t)|\, ds.$$

So for any $\tau \geq \tau_0$,

$$||\Omega_{x_0}^{t_0}||_{\text{op}} \leq \varepsilon \int_{x_0}^\infty e^{t_0 \tau} K_1(\tau) e^{-s}\, ds = \varepsilon e^{t_0 \tau} K_1(\tau) e^{-x_0}.$$

Choose τ_2 so that $\tau_2 \geq \tau_0$, and $K_1(\tau) \leq 1/2$ whenever $\tau \geq \tau_2$. Next choose ε so that $0 < \varepsilon < 1$ and $\varepsilon e^{t_0\tau_2 - x} K_1(\tau_2) < 1$. For any $x \geq 0$ and $t > 0$ we now get

$$\|\Omega_x^t\|_{\mathrm{op}} \leq \varepsilon e^{t\tau_2 - x} K_1(\tau_2) < \varepsilon e^{t\tau_2 - x}/2.$$

Note that

$$\varepsilon e^{t\tau_2 - x}/2 < 1 \quad \text{iff} \quad t < \frac{\ln 2 + \ln(1/\varepsilon) + x}{\tau_2}.$$

Let

$$h(x) := \frac{\ln 2 + \ln(1/\varepsilon) + x}{\tau_2} \quad \text{for all } x \geq 0.$$

We now extend the definition of h to $\mathbf{R}^-$. Choose a sequence $(\tilde{\tau}_0)_0^\infty$ such that

$$\tilde{\tau}_0 \geq \max\{\tau_0, \tau_1, \tau_2\}, \qquad \tilde{\tau}_n \geq \tilde{\tau}_{n-1} \quad \text{for all } n,$$

and

$$K_2(\tau) \leq 1/4(n+1) \quad \text{whenever } \tau \geq \tilde{\tau}_n.$$

Consider a negative x. There is a nonnegative integer n such that $-(n+1) \leq x < -n$. For this x and any $t > 0$

$$\|\Omega_x^t\|_{\mathrm{op}} \leq \varepsilon e^{t\tau}(|x| K_2(\tau) + K_1(\tau))$$

for any $\tau \geq \tilde{\tau}_n$. Thus

$$\|\Omega_x^t\|_{\mathrm{op}} \leq \varepsilon e^{t\tilde{\tau}_n}(3/4).$$

Note that

$$\varepsilon e^{t\tilde{\tau}_n}(3/4) < 1 \quad \text{iff} \quad t < \frac{\ln(3/4) + \ln(1/\varepsilon)}{\tilde{\tau}_n}.$$

We can now define

$$h(x) := \frac{\ln(3/4) + \ln(1/\varepsilon)}{\tilde{\tau}_n} \quad \text{if } -(n+1) \leq x < -n.$$

Thus we have define h on all $\mathbf{R}$.

It is easy to verify that $h(x) > 0$ for all x, that h is nondecreasing, and that $h(x) \to +\infty$ at least linearly as $x \to +\infty$. Let

$$\mathscr{U}_1 := \{(x, t) : 0 < t < h(x),\ x \in \mathbf{R}\}.$$

We have built in the properties that $(x_0, t_0) \in \mathscr{U}_1$ and $\|\Omega_x^t\|_{\mathrm{op}} < 1$ whenever $(x, t) \in \mathscr{U}_1$.

Proof in the case that $x_0 < 0$. Choose a positive integer N such that $-(N+1) \leq x_0 < -N$. We know that

$$\|\Omega_{x_0}^{t_0}\|_{\mathrm{op}} \leq \varepsilon e^{t_0\tau}\{|x_0| K_2(\tau) + K_1(\tau)\}$$

for all $\tau \geq \max\{\tau_0, \tau_1\}$. Choose τ_* so $\tau_* \geq \max\{\tau_0, \tau_1\}$ and $(N+1)K_2(\tau) + K_1(\tau) \leq 3/4$ for all $\tau \geq \tau_*$. Choose ε so $0 < \varepsilon < 1$ and $\varepsilon e^{t_0\tau_*}\{(N+1)K_2(\tau_*) + K_1(\tau_*)\} < 1$. Suppose $x \geq 0$ and $t > 0$. Then

$$\|\Omega_x^t\|_{\text{op}} \leq \varepsilon e^{t\tau} K_1(\tau) e^{-x} \leq \varepsilon 3 e^{t\tau - x}/4$$

whenever $\tau > \tau_*$. Note that

$$\varepsilon e^{t\tau_* - x}(3/4) < 1 \quad \text{iff} \quad t < \frac{\ln(4/3) + \ln(1/\varepsilon) + x}{\tau_*}.$$

Thus we define

$$h(x) := \frac{\ln(4/3) + \ln(1/\varepsilon) + x}{\tau_*} \quad \text{for } x \geq 0.$$

Choose a sequence $(\tau_{*n})_0^\infty$ so that $\tau_{*_0} \geq \tau_*$, $\tau_{*n} \leq \tau_{*n+1}$ for all $n \geq 0$, and $(n+1)K_2(\tau) + K_1(\tau) \leq 3/4$ whenever $\tau \geq \tau_{*n}$. Recall that

$$\|\Omega_x^t\|_{\text{op}} \leq \varepsilon e^{t\tau}(|x|K_2(\tau) + K_1(\tau))$$

whenever $\tau \geq \tau_*$. Suppose $x < 0$ and $t > 0$. Pick the positive integer n such that $-(n+1) \leq x < -n$. Then

$$\|\Omega_x^t\|_{\text{op}} \leq \varepsilon e^{t\tau_{*n}}(3/4).$$

Note that

$$\varepsilon e^{t\tau_{*n}}(3/4) < 1 \quad \text{iff} \quad t < \frac{\ln(4/3) + \ln(1/\varepsilon)}{\tau_{*n}}.$$

Finally we define h on the rest of $\mathbf{R}$ by

$$h(x) := \frac{\ln(4/3) + \ln(1/\varepsilon)}{\tau_{*n}} \quad \text{for } -(n+1) \leq x < -n.$$

It is straightforward to check that this h has the desired properties. □

For (x, t) in $\mathscr{U}_1$ we can now solve (2.5) by setting

$$B_1(x, \cdot, t) := -(I + \Omega_x^t)^{-1}[\omega_x^t] \quad \text{in } L^2(\mathbf{R}^+).$$

Because of the regularity of Ω and its decay as $x \to +\infty$, one finds from (2.5) that

$$B_1(x, \cdot, t) \in L^1(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$$

and $B_1(x, y, t)$ is C^∞ on $\mathscr{V}_1 := \{(x, y, t) : y > 0, (x, t) \in \mathscr{U}_1\}$. For (x, t) in $\mathscr{U}_1$ set

$$u_1(x, t) := -B_1(x, 0, t). \tag{2.8}$$

This u_1 is clearly C^∞ in $\mathscr{U}_1$. By Theorem T we know that such a u_1 solves KdV in $\mathscr{U}_1$. It remains to show that all derivatives of u_1 converge to 0 as $t \downarrow 0$, and that u_1 is not identically zero in $\mathscr{U}_1$.

Lemma 2.8. (a) *The operator* Ω_x^t *is bounded in* $\mathscr{L}(L^\infty(\mathbf{R}^+), L^\infty(\mathbf{R}^+))$ *for any* (x, t).

(b) *For each* x_0 *there is a* $t_0 > 0$ *such that* $(I + \Omega_x^t)$ *is invertible on* $L^\infty(\mathbf{R}^+)$ *whenever* $x \geq x_0$ *for all* $0 < t < t_0$. *Furthermore,* t_0 *can be chosen so that*

$$\|(I + \Omega_x^t)^{-1}\|_{\mathrm{op}, L^\infty(\mathbf{R}^+)} < 10$$

for such (x, t).

Proof. (a) It suffices to note that

$$\sup_{y \geq 0} \left| \int_{z=0}^{\infty} \Omega(x + y + z, t) g(z)\, dz \right| \leq \sup_{z \geq 0} |g(z)| \int_{s=x}^{\infty} |\Omega(s, t)|\, ds.$$

(b) By (a) we have

$$\|\Omega_x^t\|_{\mathrm{op}, L^\infty(\mathbf{R}^+)} \leq \int_{s=z}^{\infty} |\Omega(s, t)|\, ds.$$

By (2.4a), we may pick t_0 so that $\int_{s=x_0}^{\infty} |\Omega(s, t)|\, ds < 9/10$ when $0 \leq t \leq t_0$. Thus $(I + \Omega_x^t)^{-1}$ has a convergent Neumann series in $L^\infty(\mathbf{R}^+)$ and

$$\|(I + \Omega_x^t)^{-1}\|_{\mathrm{op}, L^\infty(\mathbf{R}^+)} \leq 10$$

for $x \geq x_0$ and $0 \leq t \leq t_0$. □

Lemma 2.9. *Let* $\nu \in \{0, 1, 2, \ldots\}$ *and* $x_0 \in \mathbf{R}$. *Then*

$$\lim_{t \downarrow 0} \partial_x^\nu u_1(x, t) = 0 \equiv \partial_x^\nu u_1(x, 0)$$

uniformly on $[x_0, +\infty)$.

Proof. Fix x_0. Because of (2.8) it suffices to show that

$$\lim_{t \downarrow 0} \partial_x^n B_1(x, 0, t) = \partial_x^n B_1(x, 0, 0) \equiv 0 \tag{2.9}$$

for all $n \geq 0$, uniformly in $[x_0, +\infty)$. Note that $\Omega(s, 0) \equiv 0$, so (2.5) yields $B_1(x, 0, 0) \equiv 0$ and $\partial_x^n B_1(x, 0, 0) \equiv 0$. Recall that

$$B_1(x, \cdot, t) = -(I + \Omega_x^t)^{-1}[\omega_x^t]$$

for (x, t) in $\mathscr{U}_1$. For $x \geq x_0$ and for sufficiently small t, we get

$$\|B_1(x, \cdot, t)\|_{L^\infty(\mathbf{R}^+)} \leq 10\|\omega_x^t\|_{L^\infty(\mathbf{R}^+)} \leq 10 \sup_{s \geq x_0} |\Omega(s, t)|.$$

By (2.4a) $B_1(x, \cdot, t) \to 0$ in $L^\infty(\mathbf{R}^+)$ uniformly in $x \geq x_0$. This proves (2.9) with $n = 0$.

Suppose (2.9) holds for all $n \leq N$, uniformly in $x \geq x_0$; we must prove (2.9) with $n = N + 1$. By (2.5)

$$\partial_x^{N+1} B_1(x, y, t) + \partial_1^{N+1} \Omega(x + y, t) + \sum_{n=0}^{N+1} \binom{N+1}{n} \Psi_n(x, y, t),$$

where ∂_1 denotes the derivative with respect to the first argument, and

$$\Psi_n := \int_0^\infty \partial_1^{N+1-n}\Omega(x+y,t)\partial_x^n B_1(x,z,t)\,dz.$$

Thus

$$\partial_1^{N+1}B_1(x,\cdot,t) = -(I+\Omega_x^t)^{-1}\left(\partial_1^{N+1}\Omega(x+[\cdot],t)+\sum_0^N \binom{N+1}{n}\Psi_n(x,\cdot,t)\right),$$

whence

$$\|\partial_x^{N+1}B_1(x,\cdot,t)\|_{L^\infty(\mathbf{R}^+)} \le 10\|\partial_1^{N+1}\Omega(x+[\cdot],t)\|_{L^\infty(\mathbf{R}^+)} + 10\sum_0^N \binom{N+1}{n}\|\Psi_n(x,\cdot,t)\|_{L^\infty(\mathbf{R}^+)}$$

for all $x \ge x_0$ and for all sufficiently small t. By Lemma 2.6 the first norm on the right goes to zero as $t \downarrow 0$. Now

$$\begin{aligned}\|\Psi_n(x,\cdot,t)\|_{L^\infty(\mathbf{R}^+)} &\le \|\partial_x^n B_1(x,\cdot,t)\|_{L^\infty(\mathbf{R}^+)}\int_0^\infty \left|\partial_1^{N+1}\Omega(x+y+z,t)\right| dz \\ &\le \|\partial_x^n B_1(x,\cdot,t)\|_{L^\infty(\mathbf{R}^+)}\|\partial_1^{N+1-n}\Omega(\cdot,t)\|_{L^1([x_0,\infty))}.\end{aligned}$$

Since $n \le N$, these terms go to zero by the induction hypothesis and Lemma 2.6. Thus (2.9) holds uniformly on $x \ge x_0$ when $n = N+1$. □

Lemma 2.10. *The function* u_1 *defined by* (2.8) *is not identically zero in* $\mathscr{U}_1$.

Proof. It follows from (2.5) that B_1 satisfies

$$B_{xx}(x,y,t) - B_{xy}(x,y,t) = u_1(x,t)B(x,y,t)$$

for $y \ge 0$ and (x,t) in $\mathscr{U}_1$. Suppose now that $u_1(x,t) \equiv 0$ in $\mathscr{U}_1$. Then B_1 solves a linear partial differential equation which has general solution

$$B_1(x,y,t) = \varphi_1(x+y,t) + \varphi_2(y,t).$$

But the assumption $u_1 \equiv 0$ together with (2.8) forces

$$0 = \partial_x B_1(x,0,t)\varphi_1'(x,t) \quad \text{in } \mathscr{U}_1,$$

where $(')$ denotes the derivative with respect to the first argument. Thus $\varphi_1(x,t) = \varphi_3(t)$ and

$$B_1(x,y,t) = \varphi_3(t) + \varphi_2(y,t) =: \psi(y,t).$$

Assume for the moment that

$$\lim_{x\to+\infty} B_1(x,y,t) = 0 \quad \text{for all } y, \text{ and all small enough } t. \tag{2.10}$$

Then it would follow that $\psi(y,t) \equiv 0$ and that $B_1(x,y,t) \equiv 0$ in $\mathscr{V}_1$. But then the Marchenko equation (2.5) yields $\Omega(x+y,t) \equiv 0$, which contradicts (2.2g).

It remains to verify (2.10). The definition of B_1 and Lemma 2.8 tells us that

$$||B_1(x\,,\cdot\,,t)||_{L^\infty(\mathbf{R}^+)} \leq 10||\Omega(\cdot\,,t)||_{L^\infty(\mathbf{R}^+)}$$

for all $x \geq 0$ and all small enough t. Lemma 2.2 now yields

$$||B_1(x\,,\cdot\,,t)||_{L^\infty(\mathbf{R}^+)} \leq \varepsilon 10 K_1(\tau)e^{t\tau - x}.$$

Fixing y and t, we get

$$|B_1(x\,,y\,,t)| \leq (\text{constant})e^{-x} \tag{2.11}$$

and (2.10) surely follows. □

This concludes the proof of Theorem 2.1.

3. Possible behavior of u_1 as x moves left

Since u_1 is a smooth function of rapid decrease as $x \to +\infty$, the misbehavior as $|x| \to \infty$ usually associated with nonuniqueness in a characteristic initial value problem must occur as x moves left. The region $\mathscr{U}_1$ where u_1 is defined does not necessarily include any strip

$$\mathscr{B}_T = \{(x\,,t) : x \in \mathbf{R}\,,\ 0 < t < T\}$$

with $T > 0$. We do not know whether u_1 can be extended to a s solution of (1.1), (1.2) on any strip $\mathscr{B}_T$. Thus the expected misbehavior may be that u_1 cannot be extended to any such strip. For the rest of this section we assume that u_1 can be so extended, say to $u^\#$, and discuss what follows about $u^\#$ from the various uniqueness results known for KdV.

The first point to notice is that the L^2 norm of $u^\#$ is not a conserved quantity: it is zero at $t = 0$, but not zero later. Indeed we will see that $u^\#$ does not evolve in $L^2(\mathbf{R})$.

Lax [11], in 1968, argued by classical methods that (1.1), (1.2) has at most one solution evolving in the class $\mathscr{C}$ of functions $v(x\,,t)$ such that

(3.1a) $\partial_x^m \partial_t^n v(x\,,t)$ is continuous for $m + 3n \leq 3$,

(3.1b) $v(x\,,t)$ and $v_{xx}(x\,,t)$ approach 0 as $x \to \pm\infty$ [fast enough that v and v_x and v_x are in $L^2(\mathbf{R})$].

Let $z(x\,,t)$ denote the constant zero solution of KdV. Clearly $z \in \mathscr{C}$, and $z(x\,,0) = 0 = u^\#(x\,,0)$. Therefore $u^\#$ does not evolve in $\mathscr{C}$. Since $u^\#$ and its derivatives decay exponentially as $x \to +\infty$, one must conclude that $u^\#$ is badly behaved as $x \to -\infty$ in the sense that (3.1a) or (3.1b) must fail as $x \to -\infty$.

Temam [14], in 1969, considered the problem (1.1), (1.2) with periodic initial data. $u_0(x + 1) = u_0(x)$. He showed uniqueness for solutions evolving in the Sobolev space $H^2((0\,,1))$. His proof remains valid for the Sobolev space $H^2(\mathbf{R})$. It follows that $u^\#$ does not evolve in $H^2(\mathbf{R})$.

In 1972, Menikoff [12] showed uniqueness for classical solutions of KdV in a class allowing linear growth at infinity, namely the class of functions $v(x,t)$ such that $\partial_x^n v(x,t) = O(|x|^{1-n})$ as $x \to \pm\infty$ for $n \in \{0,1,2,\dots,7\}$. Since $u^\#$ decays rapidly to the right, $u^\#$ may grow faster than linearly as $x \to -\infty$, or $\partial_x u^\#$ may be unbounded, or some higher derivative may fail to decay fast enough. Later we suggest that $u^\#$ probably does blow up as x moves left.

S. Kruzhkov and A. Faminskii have a very general uniqueness theorem for generalized solutions of KdV [10]. Let $\mathscr{K}_T$ denote the class of functions $v(x,t)$ such that

$$\operatorname*{ess\,sup}_{0<t<T}\left[\int_{-\infty}^{\infty}|v(x,t)|^2\,dx + \int_0^{\infty} x^{3/2}|v(x,t)|\,dx\right] < \infty.$$

Kruzhkov and Faminskii show that if u_1 and u_2 are generalized solutions of KdV which belong to $\mathscr{K}_T$, then

$$\int_{-\infty}^{\infty}\min\{e^x, 1\}|u_1(x,t) - u_2(x,t)|^2\,dx \le \gamma_T\int_{-\infty}^{\infty}|u_1(x,0) - u_2(x,0)|^2\,dx$$

for almost all t in $0 < t < T$, where γ_T is some constant.

Again let $z(x,t)$ denote the zero solution to KdV. Clearly $z \in \mathscr{K}_T$ and $z(x,0) \equiv 0 \equiv u^\#(x,0)$. So $u^\#$ cannot be in $\mathscr{K}_T$ for any T. Since $u^\#$ decays rapidly as $x \to +\infty$, one concludes that $u_1(\cdot,t)$ is not in $L^2(\mathbf{R})$.

It would be interesting to know if the analysis in [10] allows one to conclude not only that

$$\int_{-\infty}^{0}|u_1(x,t)|^2\,dx = \infty$$

but also that

$$\int_{-\infty}^{0}e^x|u_1(x,t)|^2\,dx = \infty.$$

Finally one might try to get direct control over $u^\#$ as $x \to -\infty$ by studying the nontrivial solution $\Omega^\#$ of the Airy equation. Recall that $\Omega^\#$ had the properties (1.4). Gel′fand and Shilov [7] have shown that the problem

$$\Omega_t + \Omega_{xxx} = 0, \qquad \Omega(x,0) = 0 \tag{3.2}$$

has a unique solution in any class $U(C,b)$ of measurable functions satisfying

$$|f(x,t)| \le C\exp(b|x|^{3/2}) \quad \text{for all } (x,t).$$

This is a special case of their Theorem 1 on page 42 of [7]. The function $\Omega^\#$ defined by (2.1) must therefore not belong to any $U(C,b)$. Indeed since all $\partial_x^\nu\Omega^\#$ also satisfy (3.2) none of these derivatives can belong to any $U(C,b)$. But all $\partial_x^\nu\Omega^\#$ decay fast as $x \to +\infty$. So for all ν

$$\limsup_{x\to-\infty}|\partial_x^\nu\Omega^\#(x,t)\exp(-b|x|^{3/2})| = +\infty.$$

By stationary phase arguments one sees that $\Omega^\#$ and $\partial_x\Omega^\#$ are subject to growing oscillations as $x \to -\infty$.

Unfortunately it is not easy to transfer this information about $\Omega^\#$ to the analysis of the Marchenko equation

$$B(x,y,t) + \Omega(x+y,t) + \int_0^\infty \Omega(x+y+z,t)B(x,z,t)\,dz = 0, \tag{3.3}$$

where the kernel is given by $\Omega = \varepsilon\Omega^\#$ as in §2. Let us suppose now not only that u_1 extends to $u^\#$ in the strip $\mathscr{B}_T$, but that (3.3) can be solved in $\mathscr{B}_T$ and $u^\#(x,t) = -\partial_x B(x,0,t)$ there. Thus

$$\begin{aligned} &-u^\#(x,t) + \varepsilon\partial_x\Omega^\#(x,t) + \int_0^\infty \varepsilon\partial_x\Omega^\#(x+z,t)B(x,z,t)\,dz \\ &\qquad + \int_0^\infty \varepsilon\Omega^\#(x+z,t)\partial_x B(x,z,t)\,dz = 0, \end{aligned}$$

whence

$$\begin{aligned} |u^\#(x,t) - \partial_x\Omega^\#(x,t)| &\le \int_0^\infty |\partial_x\Omega^\#(x,t)B(x,z,t)|\,dz \\ &\quad + \int_0^\infty |\Omega^\#(x+z,t)\partial_x B(x,z,t)|\,dz \\ &\le \sup_{s\ge x}|\partial_x\Omega^\#(s,t)|\cdot\|B(x,\cdot,t)\|_{L^1(\mathbf{R}^+)} \\ &\quad + \sup_{s\ge x}|\Omega^\#(s,t)|\cdot\|\partial_x B(x,\cdot,t)\|_{L^1(\mathbf{R}^+)}. \end{aligned}$$

We know that $\sup_{s\ge x}|\partial_s\Omega^\#(s,t)|$ and $\sup_{s\ge x}|\Omega^\#(s,t)|$ grow rapidly as $x \to -\infty$, but so do the available bounds on

$$\|B(x,\cdot,t)\|_{L^1(\mathbf{R}^+)} \quad\text{and}\quad \|\partial_x B(x,\cdot,t)\|_{L^1(\mathbf{R}^+)}.$$

As a result we can suspect that $u^\#(x,t)$ is subject to growing oscillations as $x \to -\infty$, but not prove it using this approach.

4. Nonuniqueness for nontrivial initial profile

Here we consider the KdV problem

$$u_t - 6uu_x + u_{xxx} = 0, \tag{4.1}$$

$$u(x,0) = U_1(x) \tag{4.2}$$

under the assumptions

(4.3a) U_1 has sufficient decay to permit the construction of a solution of (4.1), (4.2) by the classical inverse scattering method,

(4.3b) U_1 is small enough, in a sense to be made precise later.

For example, for (4.3a) it suffices that u_1 belong to $L_1^1(\mathbf{R}) \cap L_N^1(\mathbf{R}^+)$ with $N \ge 11/4$. More generally, to ensure (4.3a) one could take the initial condition

$u(x, 0) = \mu(x)$, where μ is a measure such that $(1 + |x|)^N$ is integrable with respect to $d|\mu|(x)$ for $N \geq 4$ [9]. The condition on the size of U_1 will be expressed in terms of the scattering data of the Schrödinger equation

$$-y'' + U_1(x)y = k^2 y. \tag{4.4}$$

The solution of (4.1), (4.2) guaranteed by (4.3a) is given by

$$u_1(x, t) = -\partial_x B(x, 0, t),$$

where B is determined by the Marchenko equation

$$B(x, y, t) + \Omega(x + y, t) + \int_0^\infty \Omega(x + y + z, t) B(x, z, t)\, dz = 0 \tag{4.5.1}$$

in which the kernel Ω is a particular function Ω_1 constructed from the scattering data of the Schrödinger equation (4.4). One can verify that Ω_1 solves $\Omega_t + \Omega_{xxx} = 0$. The precise assumption on the size of U_1 is

(4.3b) U_1 is small enough that $\|\Omega_1(\cdot, 0)\|_{L^1(\mathbf{R})} = 1 - \varepsilon$ for an ε with $0 < \varepsilon < 1$.

Let $\Omega_2 = \Omega_1 + \varepsilon\Omega^{\#}$, where $\Omega^{\#}$ is the special nontrivial solution of the Airy equation considered in §2 and ε is given by (4.3b). We wish to solve

$$B(x, y, t) + \Omega_2(x + y, t) + \int_0^\infty \Omega_2(x + y + z, t) B(x, z, t)\, dz = 0. \tag{4.5.2}$$

For this we need to invert $(I + \Omega^t_{1x} + \varepsilon\Omega^{\#t}_x)$. For all x,

$$\|\Omega^0_{1x}\|_{\text{op}} \leq \int_x^\infty |\Omega_1(s, 0)|\, ds \leq 1 - \varepsilon$$

and

$$\|(I + \Omega^0_{1x})^{-1}\|_{\text{op}} \leq 1/\varepsilon := \mathscr{A}.$$

Now because of the behavior of Ω_1 as $x \to +\infty$ and as $t \downarrow 0$, we find that for each $n \in \{0, 1, 2, \ldots\}$ there is a T_n such that

$$\|(I + \Omega^t_{1x})^{-1}\|_{\text{op}} \leq 2\mathscr{A}$$

whenever $x \geq -n$ and $0 \leq t \leq T_n$. We may assume that the T_n are decreasing. Let

$$\mathscr{R}_1 := \bigcup_0^\infty \{(x, t) : x > -n \text{ and } 0 \leq t \leq T_n\}.$$

Following earlier arguments we can find that $\mathscr{R}_2 := \{(x, t) : 0 < t < H(x)\}$ for some positive nondecreasing H small enough that $\|\varepsilon\Omega^{\#t}_x\|_{\text{op}} < 1/4\mathscr{A}$ in $\mathscr{R}_2$. Thus, in $\mathscr{R} := \mathscr{R}_1 \cap \mathscr{R}_2$,

$$\| -\varepsilon\Omega^{\#t}_x (I + \Omega^t_{1x})^{-1}\|_{\text{op}} \leq 1/2,$$

$$(I + \Omega^t_{2x})^{-1} = (I + \Omega^t_{1x} + \varepsilon\Omega^{\#t}_x)^{-1} = (I + \Omega^t_{1x})^{-1} \sum_{\nu=0}^\infty [-\varepsilon\Omega^{\#t}_x (I + \Omega^t_{1x})^{-1}]^\nu,$$

and

$$||(I+\Omega^t_{2x})^{-1}||_{\text{op}} \le 2\mathscr{A}\sum_0^\infty(\varepsilon/2)^\nu = 2\mathscr{A}/(1-\varepsilon/2) \le 4\mathscr{A}\,.$$

Now we can set

$$u_2(x\,,t) = -\partial_x B_2(x\,,0\,,t) \quad \text{in } \mathscr{R} = \mathscr{R}_1 \cap \mathscr{R}_2\,,$$

where B_2 solves (4.5.2). Since Ω_2 still solves $\Omega_t + \Omega_{xxx}$, u_2 still solves KdV. Because $\Omega_2(x\,,0) = \Omega_1(x\,,0) + 0$, u_2 and u_1 both satisfy (4.2). However, since $\Omega_2 \neq \Omega_1$ in $\mathscr{R}$, we expect that $u_2 \neq u_1$ in $\mathscr{R}$.

Theorem 4.1. *Under assumption* (4.3) *there exist two distinct classical solutions* $u_1(x\,,t)$ *and* $u_2(x\,,t)$ *of* (4.1), (4.2) *in the region* $\mathscr{R}$.

Proof. We prove somewhat more by showing that the functions u_1 and u_2 constructed above cannot be identical on any region of the form

$$Q(X\,,T) = \{(x\,,t): X < x < \infty\,,\ 0 < t < T\}$$

with $(X\,,T)$ in $\mathscr{R}$, and T less than the t_1 of (2.2d) or (2.2e).

Suppose that $u_1 \equiv u_2$ in $Q(X\,,T)$. Write $\mathbf{u} \equiv u_1 \equiv u_2$ there. Then from the Marchenko equations

$$B_2(x\,,y\,,t) + \Omega_2(x+y\,,t) + \int_0^\infty \Omega_2(x+y+z\,,t)B_2(x\,,z\,,t)\,dz = 0 \tag{4.6}$$

and

$$B_1(x\,,y\,,t) + \Omega_1(x+y\,,t) + \int_0^\infty \Omega_1(x+y+z\,,t)B_1(x\,,z\,,t)\,dz = 0 \tag{4.7}$$

it follows that both B_1 and B_2 satisfy

$$B_{xx}(x\,,y\,,t) - B_{xy}(x\,,y\,,t) = \mathbf{u}(x\,,t)B(x\,,y\,,t) \tag{4.8}$$

in $Q^\sim(X\,,T) = \{(x\,,y\,,t): x \ge X\,,y \ge 0\,,0 < t < T\}$ and that

$$-\partial_x B_1(x\,,0\,,t) = \mathbf{u}(x\,,t) = -\partial_x B_2(x\,,0\,,t) \quad \text{in } Q(X\,,T)\,. \tag{4.9}$$

Further by (4.2), (4.3), and the decay of Ω_1 and Ω_2 as $x \to +\infty$, one finds that $\partial_x^\nu B_j(x\,,y\,,t) \to 0$ as $x+y \to +\infty$ for any fixed t and for $\nu = 1\,,2$. Now freeze t with $0 < t < T$ and pick $x_0\,,y_0$ with $x_0 > X$, $y_0 > 0$. Integrate (4.8) over

$$R(x_0\,,y_0) = \{(x\,,y): 0 < y < y_0\,, x > x_0 + y_0 - y\}\,.$$

The result is

$$B(x_0\,,y_0\,,t) - B(x_0+y_0\,,0\,,t) = \iint_{R(x_0\,,y_0)} \mathbf{u}(x\,,t)B(x\,,y\,,t)\,dx\,dy\,.$$

By (4.9) it follows that both B_1 and B_2 satisfy

$$B(x_0\,,y_0\,,t) = \int_{x_0+y_0}^\infty \mathbf{u}(x\,,t)\,dx + \int_{y=0}^{y_0}\int_{x=x_0+y_0-y}^\infty \mathbf{u}(x\,,t)B(x\,,y\,,t)\,dx\,dy\,.$$

But this integral equation is of Volterra type and has a unique solution because of the decay of $\mathbf{u}(x, t)$ as $x \to +\infty$. Thus we see that, since $u_1 \equiv u_2$ in $Q(X, T), B_1 \equiv B_2$ in $Q^{\sim}(X, T)$. Write $\mathbf{B} = B_1 = B_2$ there. Now it follows from (4.6) and (4.7) that

$$\Omega^{\#}(x+y, t) + \int_0^{\infty} \Omega^{\#}(x+y+z, t)\mathbf{B}(x, z, t)\, dz = 0 \tag{4.10}$$

in $Q^{\sim}$ since $\Omega_2 - \Omega_1 = \varepsilon\Omega^{\#}$.

Consider the operator B_x^t in $\mathscr{L}(L^2(\mathbf{R}^+), L^2(\mathbf{R}^+))$ defined by

$$B_x^t[g](y) = \int_0^{\infty} B_2(x, z, t) g(y+z)\, dz.$$

So

$$\int_0^{\infty} \Omega^{\#}(x+[\cdot]+z, t)B_2(x, z, t)\, dz = B_x^t[\Omega^{\#}(x+[\cdot])] \quad \text{in } \mathbf{R}^+.$$

By a method analogous to the proof of (2.11) we get, for each X_0 in $\mathbf{R}$ and each positive T_0, a constant $K(T_0, X_0)$ such that

$$\|B_x^t\|_{\text{op}} \le K(T_0, X_0)e^{-x}$$

for $0 < t < T_0$ and $x > X_0$. We can now pick an X_1 such that

$$\|B_x^t\|_{\text{op}} \le 0.5 \quad \text{and} \quad \|(I + B_x^t)^{-1}\|_{\text{op}} \le 2$$

uniformly in $0 < t < T$, $x \ge X_1$. But since (4.10) says that $(I + B_x^t)\Omega^{\#} = 0$, it follows that $\Omega^{\#}(x, t) = 0$ for $x \ge X_1$ and $0 < t < T$. This contradicts the property (2.2e) of $\Omega^{\#}$. Therefore $u_2(x, t)$ cannot agree identically with $u_1(x, t)$ on any such region $Q(X, T)$. □

5. The second nonuniqueness result: global in time, but on a half-line in space

Theorem 5.1. *There are two C^{∞} soltuions u_1 and u_2 to KdV in $Q_0 = \{(x, t): x > 0, t > 0\}$ such that*

$$u_1(x, 0) \equiv u_2(x, 0) \quad \textit{for } x \ge 0,$$
$$u_1 \ne u_2 \quad \textit{in } Q_0.$$

Remark. Following the line of argument used in §4 one can show that there are many pairs of solutions v_1 and v_2 with the properties stated in Theorem 5.1. One takes a regular solution $u_3(x, t)$ of KdV evolving in Schwartz space. Using the forward problem we get a bounded solution $\Omega_3(x, t)$ satisfying the Airy equation. By adding Ω_3 to both Ω_1 and Ω_2 as constructed in the proof of Theorem 5.1, and by choosing the appropriate K in the bounds on Ω_1 and Ω_2, we can follow the same arguments to obtain new solutions v_1 and v_2 satisfying KdV in $t \ge 0$, $x \ge 0$ such that $v_1(x, 0) = v_2(x, 0) \ne u_1(x, 0) = u_2(x, 0)$.

Proof. The rest of this section is devoted to this proof. We begin by stating a lemma whose proof is deferred until after it is applied.

Lemma 5.2. *There is a constant* $K = K(\tau)$ *such that*

$$|\Omega^{\#}(x,t)| \leq Ke^{t\tau - x\tau^{1/3}}$$

for all $x \geq 0$ *and all* $t \geq 0$. *Indeed one may take*

$$K(\tau) := 2\int_{\sigma=0}^{\infty} e^{-(\sigma^2+\tau^2)^{1/3}/2}\, d\sigma .$$

In this section, as in the previous one, we consider two kernels for the Marchenko equation. This time let

$$\Omega_1(x,t) = \Omega^{\#}(x,y) + 2Ke^{t\tau - x\tau^{1/3}}, \qquad \Omega_2(x,t) = 2Ke^{t\tau - x\tau^{1/3}} .$$

Both Ω_1 and Ω_2 are C^∞ solutions of the Airy equation and have rapid decay as $x \to +\infty$. Since $\Omega^{\#}(x,0) \equiv 0$, $\Omega_1(x,0) \equiv \Omega_2(x,0)$. Further it is clear that both Ω_1 and Ω_2 are positive in the quadrant Q_0. Thus the operators $(I + \Omega^t_{1x})$ and $(I + \Omega^t_{2x})$ are positive symmetric operators on $L^2(\mathbf{R}^+)$ which can be inverted. For $i = 1, 2$ set $u_i(x,t) = -\partial_x B_i(x,0,t)$, where

$$B_i(x,y,t) + \Omega_i(x+y,t) + \int_0^\infty \Omega_i(x+y+z,t)B_i(x,z,t)\,dz = 0 .$$

By Theorem T, u_1 and u_2 both solve KdV in Q_0. Since $\Omega_1 = \Omega_2$ at $t = 0$, $B_1(x,0,t) = B_2(x,0,t)$ and $u_1 = u_2$ at $t = 0$. The proof in §4 that $u_1 \neq u_2$ in Q_0 carries over.

Thus to complete the proof of Theorem 5.1, it remains only to verify Lemma 5.2.

Proof of Lemma 5.2. For any $\tau \geq t_0$

$$\begin{aligned}\Omega^{\#}(x,t) &= \int_{i\tau-\infty}^{i\tau+\infty} e^{i\{xr(s)-ts\}} e^{-(s/i)^{2/3}}\, ds \\ &= 2e^{t\tau}\int_{\sigma=0}^{\infty} e^{-A(\sigma)} \cos(\theta(\sigma))\, d\sigma ,\end{aligned}$$

where

$$\begin{aligned}A(\sigma) &= x(\sigma^2+\tau^2)^{1/6} \sin\left(\frac{\operatorname{Arg}(\sigma+i\tau)+\pi}{3}\right) \\ &\quad + (\sigma^2+\tau^2)^{1/3} \cos\left(\frac{2\operatorname{Arg}(\tau - i\sigma)}{3}\right) \\ &= xB(\sigma) + C(\sigma) .\end{aligned}$$

The term $\theta(\sigma)$ is real; its form is unimportant. Note that $B(\sigma) \geq 0$. Thus

$$|\Omega^{\#}(x,t)| \leq 2e^{t\tau}\int_{\sigma=0}^{\infty} e^{-xB(\sigma)} e^{-C(\sigma)}\, d\sigma .$$

With $\tau > 0$ and $\sigma \geq 0$ we get

$$0 \leq \tfrac{2}{3}\operatorname{Arg}(\tau - i\sigma) < \tfrac{2}{3}\tfrac{\pi}{2} = \tfrac{\pi}{3}.$$

Thus

$$\frac{1}{2}(\sigma^2 + \tau^2)^{1/3} < (\sigma^2 + \tau^2)^{1/3} \cos\left(\frac{2\operatorname{Arg}(\tau - i\sigma)}{3}\right) < (\sigma^2 + \tau^2)^{1/3},$$

$$e^{-C(\sigma)} \leq e^{-(\sigma^2+\tau^2)^{1/3}/2},$$

and $e^{-C(\sigma)}$ is integrable over $0 < \sigma < \infty$.

Suppose we could show that $B(\sigma) \geq B(0) = \tau^{1/3}$ for all $\sigma \geq 0$. Then

$$|\Omega^{\#}(x, t)| \leq \left\{2\int_0^{\infty} e^{-C(\sigma)}\, d\sigma\right\} e^{t\tau - x\tau^{1/3}}$$

which would complete the proof of the lemma.

Lemma 5.3. *$dB(\sigma)/d\sigma > 0$ for $\sigma > 0$; $dB(\sigma)/d\sigma = 0$ for $\sigma = 0$. Thus $B(0) < B(\sigma)$ whenever $0 \leq \sigma$.*

Proof. Computation yields

$$\frac{dB(\sigma)}{d\sigma} = \frac{1}{6}\frac{2\sigma}{(\sigma^2 + \tau^2)^{5/6}} \sin\left(\frac{\operatorname{Arg}(\sigma + i\tau) + \pi}{3}\right) + (\sigma^2 + \tau^2)^{1/6} \cos\left(\frac{\operatorname{Arg}(\sigma + i\tau) + \pi}{3}\right)\frac{1}{3}\frac{d\{\operatorname{Arg}(\sigma + i\tau)\}}{d\sigma}.$$

Further computation yields

$$\frac{d}{d\sigma}\{\operatorname{Arg}(\sigma + i\tau)\} = \frac{d}{d\sigma}\left\{\arctan\left(\frac{\tau}{\sigma}\right)\right\} = \frac{-\tau}{\sigma^2 + \tau^2}.$$

Thus

$$\frac{dB(\sigma)}{d\sigma} = \frac{1}{3}\frac{1}{(\sigma^2 + \tau^2)^{5/6}}\left(\sigma \sin\frac{[\operatorname{Arg}(\sigma + i\tau) + \pi]}{3} - \tau\cos\frac{[\operatorname{Arg}(\sigma + i\tau) + \pi]}{3}\right).$$

Case 1. $\sigma \geq \tau$ (recall $\tau \geq \tau_0 > 0$). In this case

$$0 < \operatorname{Arg}(\sigma + i\tau) \leq \pi/4,$$

whence

$$\frac{\pi}{4} < \frac{\pi}{3} < \frac{\operatorname{Arg}(\sigma + i\tau) + \pi}{3} \leq \frac{\pi}{12} + \frac{\pi}{3} = \frac{5\pi}{12} < \frac{\pi}{2},$$

$$\cos\frac{[\operatorname{Arg}(\sigma + i\tau) + \pi]}{3} < \sin\frac{[\operatorname{Arg}(\sigma + i\tau) + \pi]}{3}$$

and

$$\frac{dB(\sigma)}{d\sigma} > \frac{1}{3}\frac{1}{(\sigma^2 + \tau^2)^{5/6}}\{\sigma - \tau\}\sin\frac{[\operatorname{Arg}(\sigma + i\tau) + \pi]}{3} \geq 0.$$

Case 2. $0 < \sigma < \tau$. Introduce the notation

$$D(\sigma) = \sigma\sin\frac{[\operatorname{Arg}(\sigma + i\tau) + \pi]}{3} - \tau\cos\frac{[\operatorname{Arg}(\sigma + i\tau) + \pi]}{3}.$$

Note that $D(0) = 0$. Thus

$$\frac{dB(\sigma)}{d\sigma} = \frac{1}{3}\frac{1}{(\sigma^2+\tau^2)^{5/6}}\int_0^\sigma \frac{dD(\omega)}{d\omega}\,d\omega.$$

It will suffice to show that $dD(\sigma)/d\sigma \geq 0$. Now

$$\begin{aligned}\frac{dD(\sigma)}{d\sigma} = {} & \sin\frac{[\operatorname{Arg}(\sigma+i\tau)+\pi]}{3} \\ & + \sigma\cos\frac{[\operatorname{Arg}(\sigma+i\tau)+\pi]}{3}\frac{1}{3}\frac{(-\tau)}{(\sigma^2+\tau^2)} \\ & + \tau\sin\frac{[\operatorname{Arg}(\sigma+i\tau)+\pi]}{3}\frac{1}{3}\frac{(-\tau)}{(\sigma^2+\tau^2)}.\end{aligned}$$

Since $0 \leq \sigma \leq \tau$ we get

$$\frac{5\pi}{12} \leq \frac{[\operatorname{Arg}(\sigma+i\tau)+\pi]}{3} \leq \frac{\pi}{2}$$

and

$$0 \leq \cos\frac{[\operatorname{Arg}(\sigma+i\tau)+\pi]}{3} \leq \cos\frac{5\pi}{12} < \sin\frac{5\pi}{12} \leq \sin\frac{[\operatorname{Arg}(\sigma+i\tau)+\pi]}{3}.$$

Now

$$\frac{dD(\sigma)}{d\sigma} \geq \sin\frac{[\operatorname{Arg}(\sigma+i\tau)+\pi]}{3}\left(1 - \frac{\sigma\tau+\tau^2}{3(\sigma^2+\tau^2)}\right).$$

Recalling that $0 \leq \sigma < \tau$ in this case, we get

$$1 - \frac{\sigma\tau^2+\tau^2}{3(\sigma^2+\tau^2)} \geq 1 - \frac{2\tau^2}{3(\sigma^2+\tau^2)} \geq \frac{1}{3},$$

and finally

$$\frac{dD(\sigma)}{d\sigma} \geq \frac{1}{3}\sin\frac{[\operatorname{Arg}(\sigma+i\tau)+\pi]}{3} \geq 0. \quad \square$$

References

1. A. Cohen, *Decay and regularity in the inverse scattering problem*, J. Math. Anal. Appl. **87** (1982), 395–426.
2. A. Cohen and Th. Kappeler, *Solutins of the Korteweg-deVries equation with initial profile in* $L_1^1(\mathbf{R}) \cap L_N^1(\mathbf{R}^+)$, SIAM J. Math. Anal. **18** (1987), 991–1025.
3. P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979), 121–251.
4. H. Dym and H. McKean, *Gaussian processes, function theory, and the inverse spectral problem*, Academic Press, New York, 1976.
5. L. Faddeev, *Properties of the S-matrix of the one-dimensional Schrodinger equation*, Amer. Math. Soc. Transl. **65** (1967), 129–166.
6. C. Gardner, J. Greene, M. Kruskal and R. Miura, *Method for solving the Korteweg-deVries equation*, Phys. Rev. Lett. **19** (1967), 1095–1097.
7. I. M. Gel′fand and G. E. Shilov, *Generalized functions.* vol. 3, *Theory of differential equations* (translated by M. E. Mayer), Academic Press, New York, 1967.
8. L. Hörmander, *Linear partial differential equations*, Springer-Verlag, New York, 1976.

9. Th. Kappeler, *Solutions to the Korteweg-deVries equation with irregular initial profile*, Comm. Partial Differential Equations **11** (1986), 927–945.

10. S. N. Kruzhkov and A. V. Faminskii, *Generalized solutions of Cauchy problem for the Korteweg-deVries equation*, Math. USSR-Sb. **48** (1984), 391–421.

11. P. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **25** (1968), 467–490.

12. A. Menikoff, *The existence of unbounded solutions of the Korteweg-deVries equation*, Comm. Pure Appl. Math. **25** (1972), 407–432.

13. S. Tanaka, *Korteweg-deVries equation: construction of solution from scattering data*, Osaka J. Math. **11** (1974), 49–59.

14. R. Temam, *Sur un problème non linéaire*, J. Math. Pures Appl. **49** (1969), 159–172.

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